The Mathematics Teachers Need to Know

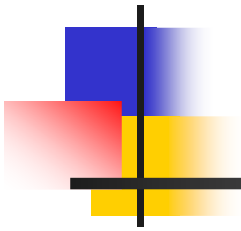
Richard James Milgram
Stanford University

Presented at the Center on Instruction
Mathematics Summit
November 13, 2006

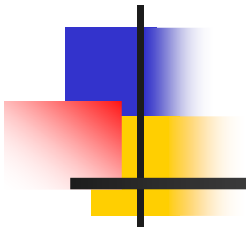
The Center on Instruction is operated by RMC Research Corporation in partnership with the Florida Center for Reading Research at Florida State University; RG Research Group; the Texas Institute for Measurement, Evaluation, and Statistics at the University of Houston; and the Vaughn Gross Center for Reading and Language Arts at the University of Texas at Austin.

The contents of this PowerPoint were developed under cooperative agreement S283B050034 with the U.S. Department of Education. However, these contents do not necessarily represent the policy of the Department of Education, and you should not assume endorsement by the Federal Government.

2006



The Mathematics Teachers Need to Know

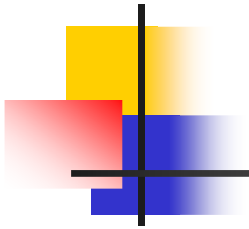


Over the past 3 years two communities - math education, mathematics - have begun to cooperate to bring the strengths of international curricula to the United States

Reaching for Common Ground in K–12 Mathematics Education

Deborah Loewenberg Ball, Joan Ferrini-Mundy, Jeremy Kilpatrick, R. James Milgram, Wilfried Schmid, and Richard Schaar

Notices of American Mathematics Society, Oct. 2005



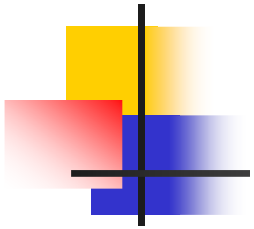
Over the past decade, much debate has arisen between mathematicians and mathematics educators. These debates have significantly distracted the attention of key players at all levels, and have impeded efforts to improve mathematics learning in this country. This document represents an attempt to identify a preliminary list of positions on which many may be able to agree.

Our effort arose out of discussions between Richard Schaar and major players in both communities. He suspected that some of these disagreements might be more matters of language and lack of communication than representative of fundamental differences of view. To test this idea, he convened a small group of mathematicians and mathematics educators.[1]

1. Basic skills with numbers continue to be vitally important for a variety of everyday uses. They also provide crucial foundation for the higher-level mathematics essential for success in the workplace which must now also be part of a basic education. Although there may have been a time when being to able to perform extensive paper-and-pencil computations mechanically was sufficient to function in the workplace, this is no longer true. Consequently, today's students need proficiency with computational procedures. *Proficiency*, as we use the term, includes both computational fluency and understanding of the underlying mathematical ideas and principles.[2]

2. Mathematics requires careful reasoning about precisely defined objects and concepts. Mathematics is communicated by means of a powerful language whose vocabulary must be learned. The ability to reason about and justify mathematical statements is fundamental, as is the ability to use terms and notation with appropriate degrees of precision. By *precision*, we mean the use of terms and symbols, consistent with mathematical definitions, in ways appropriate for students at particular grade levels. We do *not* mean formality for formality's sake.

3. Students must be able to formulate and solve problems. Mathematical problem solving includes being able to (a) develop a clear understanding of the problem that is being posed; (b) translate the problem from everyday language into a precise mathematical question; (c) choose and use appropriate methods to answer the question; (d) interpret and evaluate the solution in terms of the original problem, and (e) understand that not all questions admit mathematical solutions and recognize problems that cannot be solved mathematically.



The Key Topics in a Successful Math Curriculum

R. James Milgram
Department of Mathematics
Stanford University
Stanford, CA

Hung-Hsi Wu
Department of Mathematics
University of California, Berkeley
Berkeley, CA

Analysis of the results from TIMSS suggests that the U.S. school mathematics curriculum is a mile wide and an inch deep.[1] It covers too many topics and each topic is treated superficially. By contrast, the structure of mathematics instruction in countries which outperformed the U.S. follows a strikingly different pattern. In all cases, only a few carefully selected focus topics are taught and learned to mastery by students in the early grades. At the fourth grade level, since the students in these countries have not been exposed to as broad a curriculum as U.S. students, it sometimes appears on standardized tests such as TIMSS that they perform at a comparable level to U.S. students, but by grade eight the students in the leading countries are far outperforming our students. In fact, key test items already show serious weaknesses in our fourth grade student performances.[2] This difference becomes even greater by the end of high school, where we have a much smaller group of students matching the level of the better students in many countries, including some which are much poorer than we are. [3]

The heart of this proposed program is the creation of six volumes of special instructional materials, each volume devoted to one of the following six topics:

The California Standards come in two flavors

- General Standards –
  - Better than most, maybe a block wide
- Green dot standards

Grade One Mathematics Content Standards

By the end of grade one, students understand and use the concept of ones and tens in the place value number system. Students add and subtract small numbers with ease. They measure with simple units and locate objects in space. They describe data and analyze and solve simple problems.

Number Sense

1.0 Students understand and use numbers up to 100:

1.1 Count, read, and write whole numbers to 100.

1.2 Compare and order whole numbers to 100 by using the symbols for less than, equal to, or greater than (<, =, >).

Which of the following are correct and which are incorrect?

(a) 75 > 76 (b) 48 < 42 (c) 89 > 91 (d) 59 < 67 (e) 34 = 33

1.3 Represent equivalent forms of the same number through the use of physical models, diagrams, and number expressions (to 20) (e.g., 8 may be represented as 4 + 4, 5 + 3, 2 + 2 + 2 + 2, 10 – 2, 11 – 3).

1.4 Count and group objects in ones and tens (e.g., three groups of 10 and 4 equals 34, or 30 + 4).

A certain brand of chewing gum has 10 pieces in each pack. If there are 14 students, what is the smallest number of packs we must buy to make sure each student gets at least one piece of gum? If there are 19 students? What about 21 students?

There are 5 quarters, 9 dimes, 3 nickels, and 8 pennies. They are supposed to be put in piles of ten (coins). How many such piles can be formed by all these coins, and how many are left over?

1.5 Identify and know the value of coins and show different combinations of coins that equal the same value.

Give each student a plastic set of 25 pennies, 5 nickels, and 2 dimes. Ask the class to find different ways to make 25 cents.

2.0 Students demonstrate the meaning of addition and subtraction and use these operations to solve problems:

2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.

I had 10 cupcakes, but I ate 3 of them. How many cupcakes do I have left? How many if I had 18 and ate 5?

2.2 Use the inverse relationship between addition and subtraction to solve problems.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for grade one.

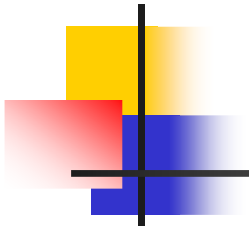
Key Standards

Number Sense	Algebra and Functions	Measurement and Geometry	Statistics, Data Analysis, and Probability	Mathematical Reasoning*
Kindergarten				
1.0 1.1 1.2 1.3 2.0 **2.1** 3.0 3.1	1.0 **1.1**	**1.0** 1.1 1.2 1.3 1.4 2.0 2.1 2.2	1.0 1.1 **1.2**	1.0 1.1 1.2 2.0 2.1 2.2
Grade One				
1.0 **1.1** **1.2** 1.3 1.4 1.5 2.0 **2.1** **2.2** **2.3** **2.4** **2.5** 2.6 2.7 3.0 3.1	1.0 1.1 1.2 1.3	1.0 1.1 1.2 2.0 2.1 2.2 2.3 2.4	1.0 1.1 1.2 2.0 **2.1**	1.0 1.1 1.2 2.0 2.1 2.2 3.0
Grade Two				
1.0 **1.1** 1.2 **1.3** 2.0 **2.1** **2.2** 2.3 **3.0** **3.1** **3.2** **3.3** 4.0 **4.1** **4.2** **4.3** 5.0 **5.1** **5.2** 6.0 6.1	1.0 **1.1** 1.2 1.3	1.0 1.1 1.2 **1.3** 1.4 1.5 **2.0** **2.1** **2.2**	**1.0** 1.1 1.2 1.3 1.4 **2.0** 2.1 2.2	1.0 1.1 1.2 2.0 2.1 2.2 3.0
Grade Three				
1.0 1.1 1.2 **1.3** 1.4 **1.5** 2.0 **2.1** **2.2** **2.3** **2.4** 2.5 2.6 2.7 2.8 3.0 3.1 **3.2** **3.3** 3.4	1.0 **1.1** 1.2 1.3 1.4 1.5 2.0 **2.1** 2.2	1.0 1.1 **1.2** **1.3** 1.4 2.0 **2.1** **2.2** **2.3** 2.4 2.5 2.6	1.0 1.1 **1.2** **1.3** 1.4	1.0 1.1 1.2 2.0 2.1 2.2 2.3 2.4 2.5 2.6 3.0 3.1 3.2 3.3

*It should be noted that the strand of mathematical reasoning is different from the other four strands. This strand, which is inherently embedded in each of the other strands, is fundamental in developing the basic skills and conceptual understanding for a solid mathematical foundation. It is important when looking at the standards to see the reasoning in all of them. Since this is the case, the key topics in the mathematical reasoning strand are not highlighted. Standards with the ● symbol are the most important ones to be covered within a grade level.

Key Standards

Number Sense	Algebra and Functions	Measurement and Geometry	Statistics, Data Analysis, and Probability	Mathematical Reasoning
Grade Four				
1.0 **1.1** **1.2** **1.3** **1.4** 1.5 1.6 1.7 **1.8** **1.9** 2.0 2.1 2.2 **3.0** **3.1** **3.2** **3.3** **3.4** 4.0 4.1 **4.2**	1.0 1.1 **1.2** **1.3** 1.4 **1.5** **2.0** **2.1** **2.2**	1.0 1.1 1.2 1.3 1.4 **2.0** **2.1** **2.2** **2.3** 3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8	1.0 1.1 1.2 1.3 2.0 2.1 2.2	1.0 1.1 1.2 2.0 2.1 2.2 2.3 2.4 2.5 2.6 3.0 3.1 3.2 3.3
Grade Five				
1.0 1.1 **1.2** 1.3 **1.4** **1.5** 2.0 **2.1** **2.2** **2.3** 2.4 2.5	1.0 1.1 **1.2** 1.3 **1.4** **1.5**	1.0 **1.1** **1.2** **1.3** 1.4 2.0 **2.1** **2.2** 2.3	1.0 1.1 1.2 1.3 **1.4** **1.5**	1.0 1.1 1.2 2.0 2.1 2.2 2.3 2.4 2.5 2.6 3.0 3.1 3.2 3.3
Grade Six				
1.0 **1.1** **1.2** **1.3** **1.4** **2.0** 2.1 2.2 **2.3** **2.4**	1.0 **1.1** 1.2 1.3 1.4 2.0 2.1 **2.2** 2.3 3.0 3.1 3.2	1.0 **1.1** 1.2 1.3 2.0 2.1 **2.2** 2.3	1.0 1.1 1.2 1.3 1.4 2.0 2.1 **2.2** **2.3** **2.4** **2.5** 3.0 **3.1** 3.2 **3.3** 3.4 **3.5**	1.0 1.1 1.2 1.3 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 3.0 3.1 3.2 3.3
Grade Seven				
1.0 1.1 **1.2** 1.3 **1.4** **1.5** 1.6 **1.7** 2.0 2.1 **2.2** **2.3** 2.4 **2.5**	1.0 1.1 1.2 **1.3** 1.4 1.5 2.0 2.1 2.2 3.0 3.1 3.2 **3.3** **3.4** **4.0** **4.1** **4.2**	1.0 1.1 1.2 **1.3** 2.0 2.1 2.2 2.3 2.4 3.0 3.1 3.2 **3.3** **3.4** 3.5 **3.6**	1.0 1.1 1.2 **1.3**	1.0 1.1 1.2 1.3 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 3.0 3.1 3.2 3.3



- In California the green dot standards are the focus of most instruction since these topics comprise 85% of the state exams in grades 2 – 7.
- They were selected by mathematicians, teachers and math educators as the key topics students needed to know.

NCTM's Focal Points Represent the Same Perspective

Here are Details

Focus Topics – Numbers

K	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5
Represent, order compare whole numbers	Addition and subtraction	Base10 number system, place value	Multiplication division	Quick recall multiplication division facts, fluency with multiplication	Understand and become fluent with whole number division
Describe shapes and spaces	Whole number relationships and grouping	Quick recall of basic addition subtraction facts	Fractions, equivalent fractions	Decimals, connection between fractions and decimals	Understand and become fluent with addition, subtraction of fractions
Order objects by measurable attributes	Composing and decomposing geometric shapes	Measuring lengths	Properties of 2 dimensional shapes	Understand area, determine area of 2 dimensional shapes	Volume and surface area for three dimensional shapes

Focus Topics Fractions

K	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5
Represent, order compare whole numbers	Addition and subtraction	Base10 number system, place value	Multiplication division	Quick recall multiplication division facts, fluency with multiplication	Understand and become fluent with whole number division
Describe shapes and spaces	Whole number relationships and grouping	Quick recall of basic addition subtraction facts	Fractions, equivalent fractions	Decimals, connection between fractions and decimals	Understand and become fluent with addition, subtraction of fractions
Order objects by measurable attributes	Composing and decomposing geometric shapes	Measuring lengths	Properties of 2 dimensional shapes	Understand area, determine area of 2 dimensional shapes	Volume and surface area for three dimensional shapes

Focus Topics Fractions, Ratios

Grade 6	Grade 7	Grade 8
Understanding and fluency multiplying and dividing fractions and decimals	Understand and apply proportionality and similarity	Analyze and represent linear equations. Solve linear equations and systems of linear equations
Connect ratio and rate to whole number multiplication and division	Understand and use formulas to determine surface area and volume	Use distance and angle to analyze figures in two and three dimensions
Write, interpret and use expressions and equations	Solve linear equations and operate with all rational numbers, including negative numbers	Analyze and summarize data sets

Focus Topics, Algebra, Data

Grade 6	Grade 7	Grade 8
Understanding and fluency multiplying and dividing fractions and decimals	Understand and apply proportionality and similarity	Analyze and represent linear equations. Solve linear equations and systems of linear equations
Connect ratio and rate to whole number multiplication and division	Understand and use formulas to determine surface area and volume	Use distance and angle to analyze figures in two and three dimensions
Write, interpret and use expressions and equations	Solve linear equations and operate with all rational numbers, including negative numbers	Analyze and summarize data sets

It is clear that we are achieving common ground on standards

There are differences in grade level but not in the view of what matters

What else do we need to do

- Dramatically improve textbooks
- Publishers are trying, but they have to deal with existing state standards
- A promising method is to translate and disseminate math texts from high achieving countries
  - Example: Russian texts besides the Grade 1 – 3 1980's texts translated by UCSMP

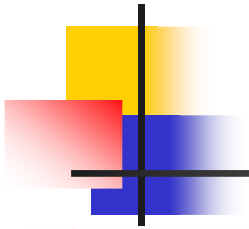
Russian Grade 4: TOC

- Integers and Fractions:
  - Numbers and Sets
  - Equalities and Inequalities
  - Equations and Inequalities
  - Addition and Subtraction
  - Multiplication and its Properties
  - Applying the Laws of Addition and Multiplication
  - Division and its Properties

Russian Grade 4: TOC

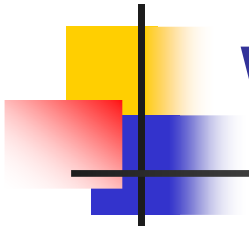
- Decimal Fractions:
  - The Decimal System of Counting and Measurement
  - Addition and Subtraction
  - Multiplication
  - Division
  - Calculations and Constructions

#1. Numbers and Sets



#3. Equations and Inequalities

#7. Division and its Properties



What else do we need to do

- Improve test development and reliability. Existing tests are filled with mathematical errors.

Problem From WA Practice Exit Exam
5 of first 8 were mathematically incorrect.

High School Mathematics WASL Practice Test

8 Study the pattern shown in the table.

What is the value of *s* when *r* equals 10?

r	0	2	4	6	8	
s	7	11	23	43	71	

Support your answer using words, numbers, and/or diagrams.

What's wrong with patterns?

After explaining to a student through various lessons and examples that:

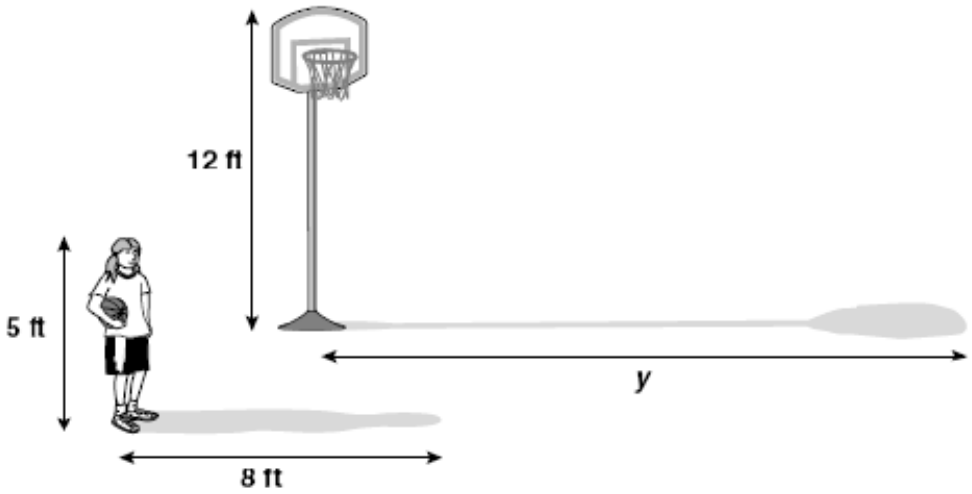
$$\lim_{x \to 8} \frac{1}{x-8} = \infty$$

I tried to check if she really understood that, so I gave her a different example. This was the result:

$$\lim_{x \to 5} \frac{1}{x-5} = \text{ɯ}$$

Some Further Examples from the WASL

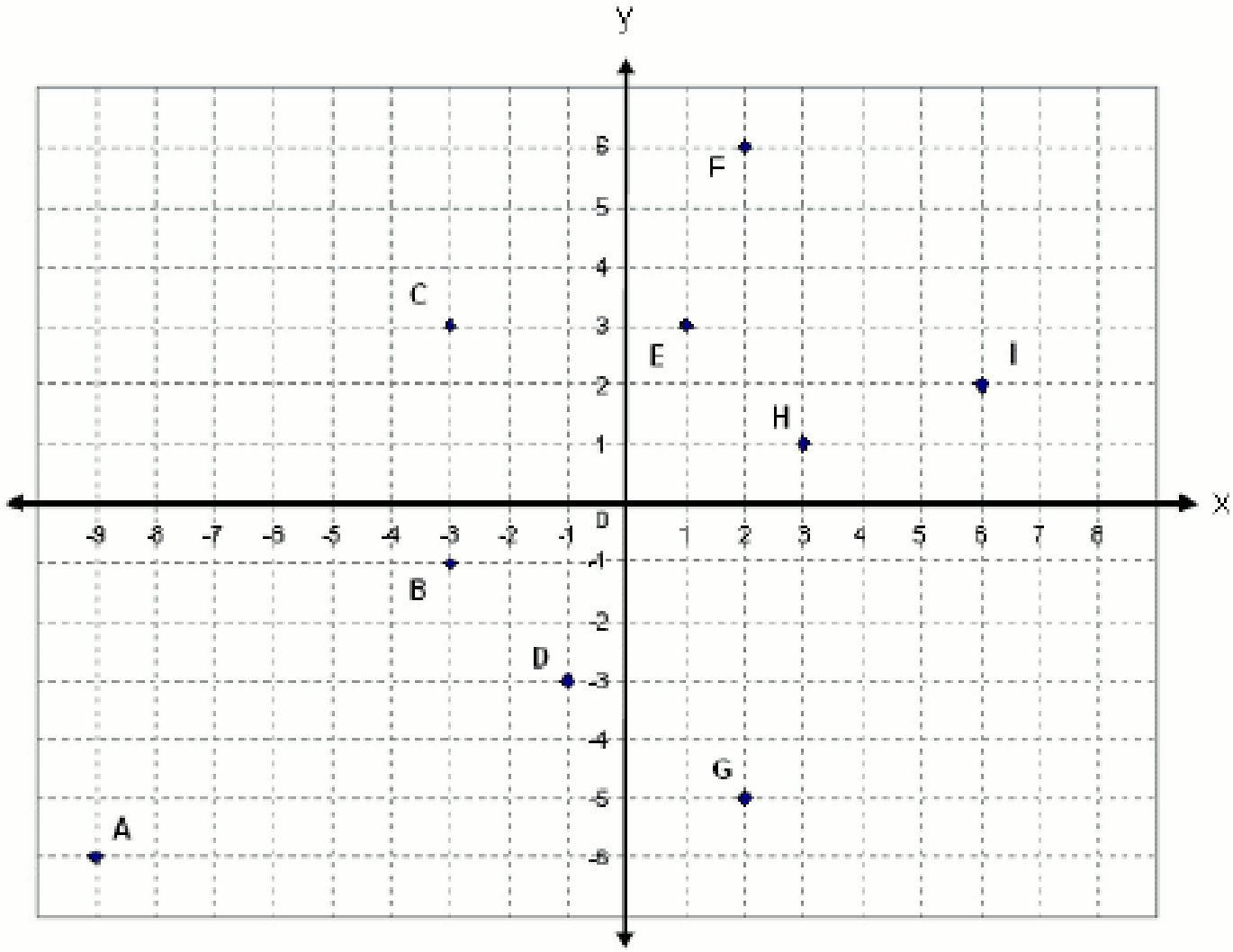
2 A 5-foot-tall person casts an 8-foot shadow. A vertical pole that supports a basketball hoop is 12 feet high.



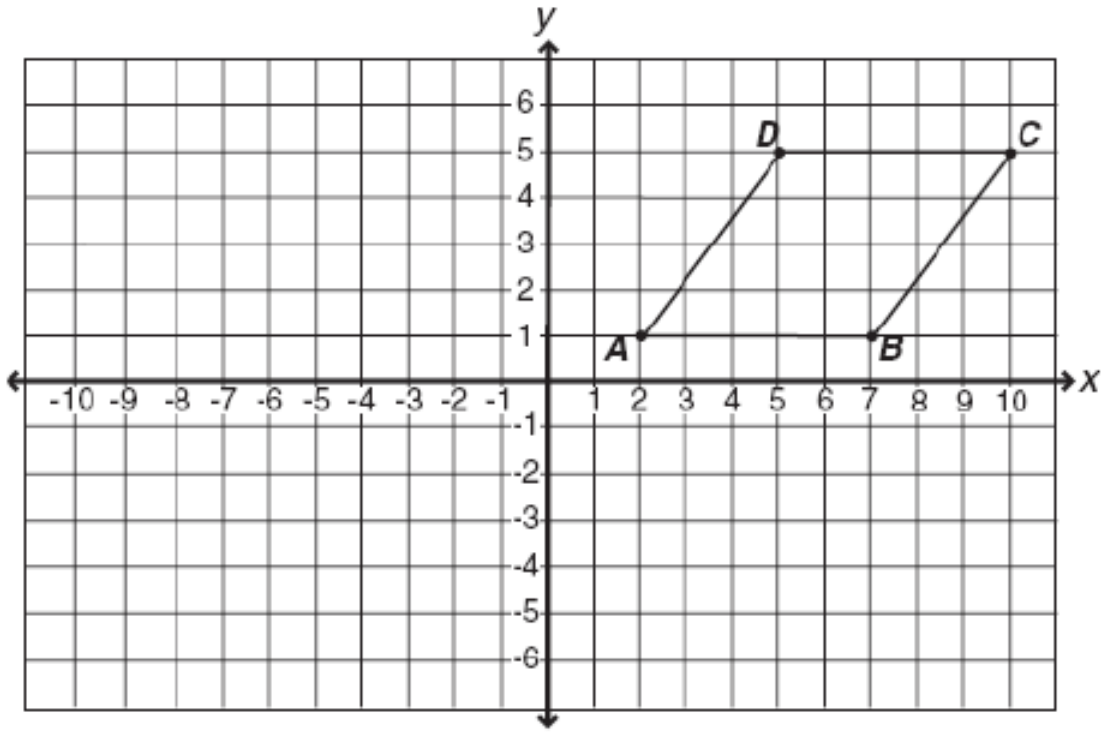
How long is the shadow of the pole?

- ○ A. 7.5 feet
- ○ B. 15 feet
- ○ C. 19.2 feet
- ○ D. 25 feet

7 On Enrique's grid find all the points whose ***x*** coordinate is one third of its ***y*** coordinate.



26 Look at the figure on the grid.



When the figure is translated so that point *C* is moved to the origin, and point *D* is moved to the *x*-axis, what are the new coordinates of point *A*?

- ○ **A.** (-9, -6)
- ○ **B.** (-4, -8)
- ○ **C.** (-3, -9)
- ○ **D.** (-8, -4)

What else do we need to do

- Create more sensible certification requirements for teachers
  - The high achieving countries have a general certification only for grades 1 – 4,
  - They have a TWO topic certification for grades 5 – 8.

Typical Requirements in High Achieving Countries: 1-4

- 1.1 Sets and logic; relations, functions, sequences
- 1.2 Elementary Number Theory
- 1.3 Mathematics Education I.
- 1.4 Mathematics Education II.
- 1.5 Teaching Geometry and Measurement
- 1.6 Combinatorics, Probability, Statistics and its instruction

Typical Requirements in High Achieving Countries: 5-8

- Algebra Semester I (number theory)
- Semester II (classical and linear algebra)
- Semester III (abstract algebra)
- Four Semesters of Analysis
- Three Semesters of Geometry
- Three courses on math methodology and teaching
- Three field work courses

What else do we need to do

- Translate and disseminate math methods material from other countries
  - Example: The First Volume of the Hungarian 1 – 3 material.

Here is the Opening of the First Hungarian Methods Text

Forming of the concept of natural numbers in pre- school and in school

When a parent takes his child to first grade he is usually happy to tell the teacher that little Tommy can count till 100 already. Pride is a good sign, because it signals: it's important for the parent to see his child develop not only physically, but intellectually as well.	Introductory thoughts
Sometimes it soon turns out (and the sooner the better) that the correctly named numbers by this same child have no content; the child knows the numerals, but not the concept of numbers. He can say the words in order: one, two, three…thirty eight, thirty nine, forty, forty one…., but he cannot really tell which one is more: 5 apples or 7 apples, or have a clear picture about the order of size, equalities or the contents of the numerals.	Is it sure that a child who can count knows the concept of numbers?
We need to add quickly that it's not a problem: **building the concept of numbers is the task of the school.** If a child doesn't possess the concept of numbers when starting school, then he will learn it with the teacher's help; it's not too late in the first grade. The only thing is that the child who is ahead, will need to do something else, than the one who is just learning. (It's not an easy job to adapt to different needs, but developing starts with learning about the different levels: it's important for the conscious developmental process.)	

1. Reality and numbers

The beginnings of concept building

The concept is built on two empirical bases	The concept of natural numbers is basically built on **two empirical bases.** On one hand, children get to know the numbers as concrete things, on the other hand as measuring numbers. The two contents are being developed side by side, completely isolated for a while, then at one point they intertwine. In spite of the two contents being isolated, we can find a nice parallel between them. The main steps of the forming of the two contents are the following:
The main steps of the building process in sketches	**COUNTING NUMBERS** **MEASURING NUMBERS** **1. Perceptual impressions, comparisons about the following relations:** More, less; taller, shorter; longer, shorter; wider, narrower

First Chapter Best Selling U.S. Math Methods Text

Second Chapter Best Selling U.S. Math Methods Text

Third Chapter Best Selling U.S. Math Methods Text

What else do we need to do

- Create course material for better pre-service training courses.

Liping Ma and I are working together at Stanford on this

The Original Course Plan

			Title	Content
4/4	1	Ed. Theory	What is mathematics for elementary teachers? (Understanding the Whole picture) Why young students learning mathematics?	
4/6	2		Three kinds of arithmetic	A collection of algorithms, Synonym of number theory, FLM
4/11	3		Two traditions of mathematics	*Elements* vs. *Nine Chapters*
4/13	4		Arithmetic as the foundation for learning mathematics	How arithmetic was regulated into a microcosm of the discipline 基本思路/变化件
4/18	5	Core Structure	The core structure of arithmetic (1) Five general definitions	Checking up knowledge "bottom line"; The five general definitions
4/20	6		The core structure of arithmetic (2) Whole number + / −	Basic format and basic terminology
4/25	7		The core structure of arithmetic (3) Whole number × / ÷	The first upgrading of the concept of "unit" 式/文/应/线
4/27	8		The core structure of arithmetic (4) Fraction: concept	The second upgrading of the concept of "unit" 式/文/应/线
5/2	9		The core structure of arithmetic (5) Fraction: + / −	The impact of the second upgrading (a) 式/文/应/线
5/4	10		The core structure of arithmetic (6) Fraction: Multiplication	The impact of the second upgrading (b) 式/文/应/线
5/9	11		***Midterm***	
5/11	12		The core structure of arithmetic (7) Fraction: Division (a)	The impact of the second upgrading (b'_1) 式/文/应/线
5/16	13		The core structure of arithmetic (8) Fraction: Division (b)	The impact of the second upgrading (b'_2) 式/文/应/线
5/18	14		Notation, Base 10, Place value (1)	The whole picture of notation
5/23	15		Notation, Base 10, Place value (2)	What is the key piece?
5/25	16		Elementary Geometry / Measurement	The consistency with the core structure of arithmetic
5/30	17		The sequential order of elementary mathematics	Two sequences (the whole one and a detailed sect of it)
6/1	18		Word problem (1)	Multiple-step/operation problems
6/6	19		Word problem (2)	"Typical problems"
6/8	20		Wrapping up: A Review of Terminology 术语	
6/13	21		***Final Exam***	

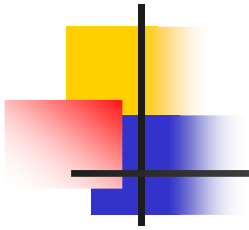
Note that we see the same structure as was evident in the Focus Topics and the Green Dot Standards

Liping Ma Explaining Place Value

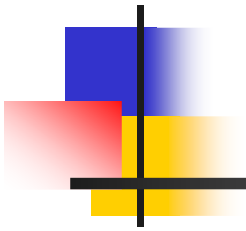
This snippet is rich with issues

- Originally, I thought what was important was the shakiness of the students' understanding of place-value.
- However, on further thought, I realized they do understand. They were, however, sensing the lack of a definition
- Liping's failure to initiate the discussion with a precise definition is critical

This snippet is rich with issues



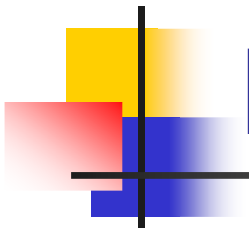
- Definitions are a key part of the signature pedagogy of mathematics, and this pedagogy is there for a reason.



Real Thought Must be Given to Mathematics Pedagogy in Pre-service Teacher Training

This is a case where one size does not fit all

- Among the issues are that for pedagogical reasons definitions play virtually no role in elementary math instruction in this country
- As an example: It is not unusual for a teacher to let students define fraction addition and multiplication for themselves via class discussion. So we often see



Fraction Addition

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

Which is closely related to