Pre-Algebra and Algebra Instruction and Assessments

CENTER ON INSTRUCTION

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2006

Algebra and
NCTM Focal Points

Overview of the Focal Points

- PreK through grade 8
- Important math topics / content in mathematics by grade
- Cohesive clusters of related ideas / concepts / skills that form foundation for higher level mathematics
  - NOT discrete topics that teachers teach and students master

Algebra Learning Expectations Across States: Grade 4

State	Algebra	Total
California	7	43
Texas	4	32
New York	5	56
Florida	10	89
Ohio	6	48
Michigan	0	56
New Jersey	6	56
North Carolina	3	26
Georgia	3	45
Virginia	2	41

Algebra in Prekindergarten

- Patterns: recognize and duplicate sequential patterns

Algebra in Kindergarten

- Identify, duplicate, and extend:
  - Number patterns
  - Sequential and growing patterns with shapes
- Preparation for creating rules that describe relationships
  - E.g., each new pattern grows by one

Algebra in Grade 1

- Understand connection between counting and operations of addition and subtraction (e.g., adding two is the same as “counting on” two)
- Use properties of commutativity and associativity to add whole numbers
- Begin to relate addition and subtraction as inverse operations

Algebra in Grade 2

- Use number patterns to extend understanding of properties of numbers and operations
  - E.g., skip counting builds foundation of understanding multiples and factors

Algebra in Grade 3

- Understand meaning of multiplication and division (equal “jumps,” equal-sized groups, area models)
- Creation and analysis of patterns and relationships involving multiplication and division

Algebra in Grade 4

- Students understand number patterns involving all operations and nonnumeric growing or repeating patterns
  - Can use rules to describe a sequence of numbers or objects
  - E.g., 2, 4, 8, 16, 32

Algebra in Grade 5

- Use patterns, models, relationships to write and solve simple equations and inequalities

  E.g., 2, 4, 8, 16, _

  _ = 16*2 = 32.
- Create graphs of simple equations

x:	1	2	3	4	5
y:	2	4	8	16	32

Algebra in Grade 6

- Algebra is a Curriculum Focal Point for the first time on its own
- Write and evaluate mathematical expressions; use expressions and formulas to solve problems.
- Expressions in different forms can be equivalent
- Understand that variables represent numbers whose precise values are not yet specified

Algebra in Grade 7

- Algebra is in all 3 Curriculum Focal Points

Algebra in Grade 8

- Analyzing and representing linear functions, solving linear equations, and systems of linear equations
- Also beginning to explore nonlinear functions

Algebra Instruction

Algebra and the Elementary Grades

- Algebra has often been characterized as developmentally constrained due to its inherent abstractness (e.g., Kieran, 1981, 1985; Vergnaud, 1985)
- Research in the former Soviet Union suggested that young children could generalize arithmetic, moving from particular to generalized numbers, learning to use variables and covariation in word problems, and focusing on the concept of function (Davydov, 1991, Bodanskii, 1991)

Algebra and the Elementary Grades

- Recent research suggests that inappropriate instruction may have had a decisive role in the poor results from early studies of algebraic reasoning among adolescents in the U.S. (Booth, 1988; Schliemann & Carraher, 2002).

Critical Topics for Teaching and Learning Algebra

1. Variables and constants
2. Decomposing and setting up word problems
3. Symbolic manipulation
4. Functions
5. Inductive reasoning and mathematical induction

Milgram (2005)

Connecting Arithmetic to Algebra

Numerical Expression	Algebraic Expression
7 + 3	n + 3

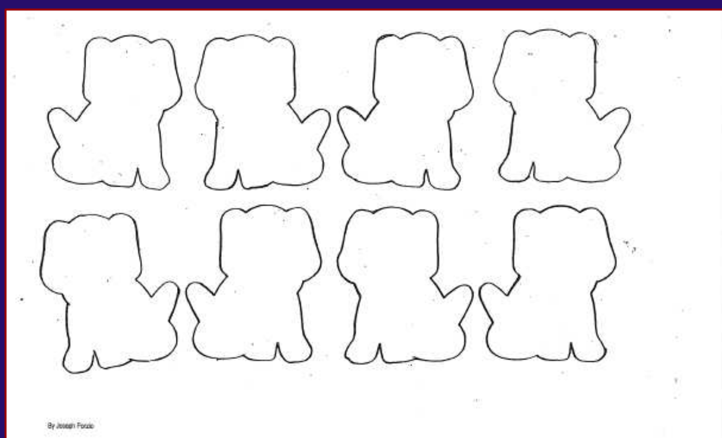
variable

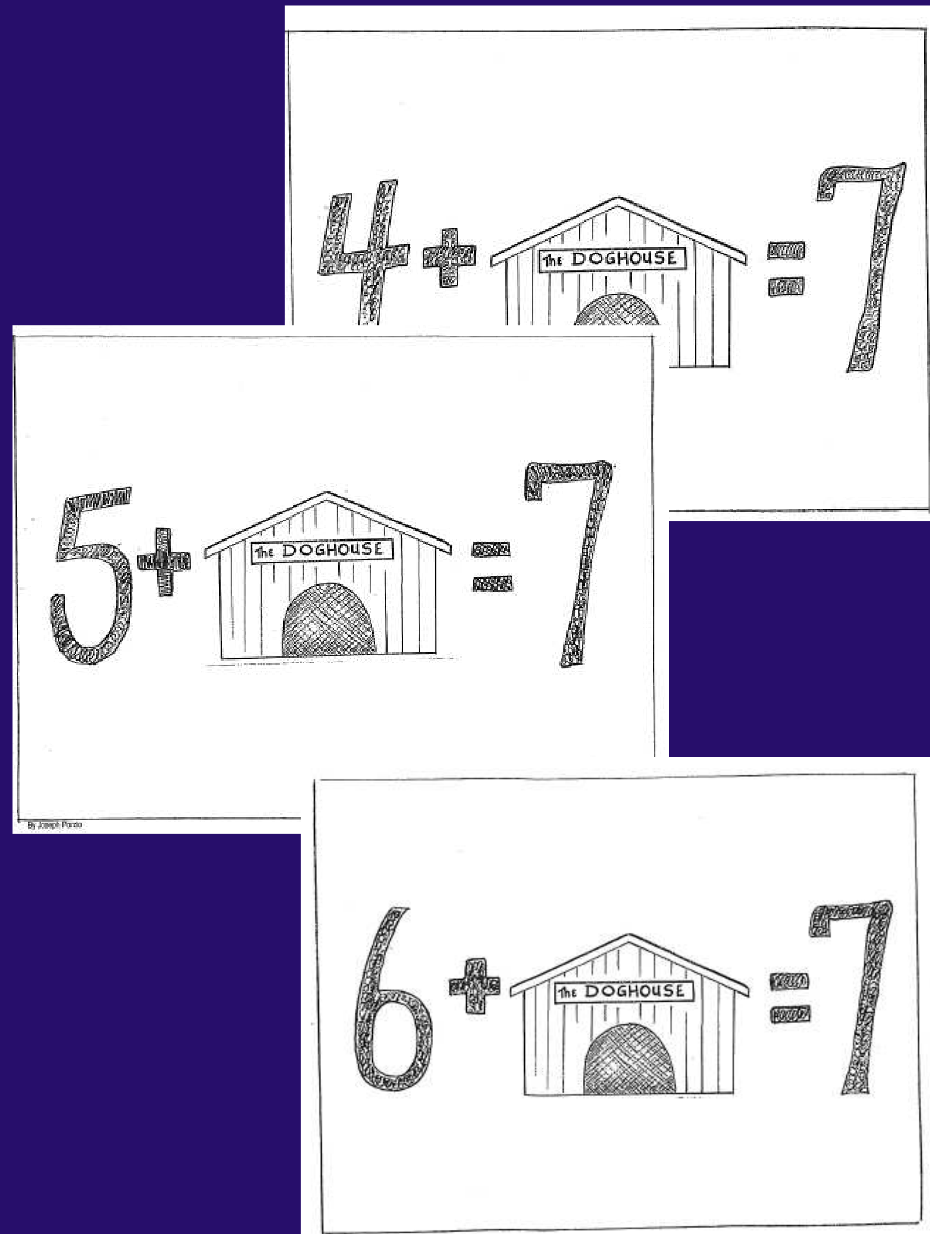
constant

Here is how you read algebraic expressions

sum	n + 3 “n plus 3”	difference	n – 3 “n less 3”
product	3 x n or 3n “3 times n”	quotient	n÷3 “n divided by 3”

In the Dog House

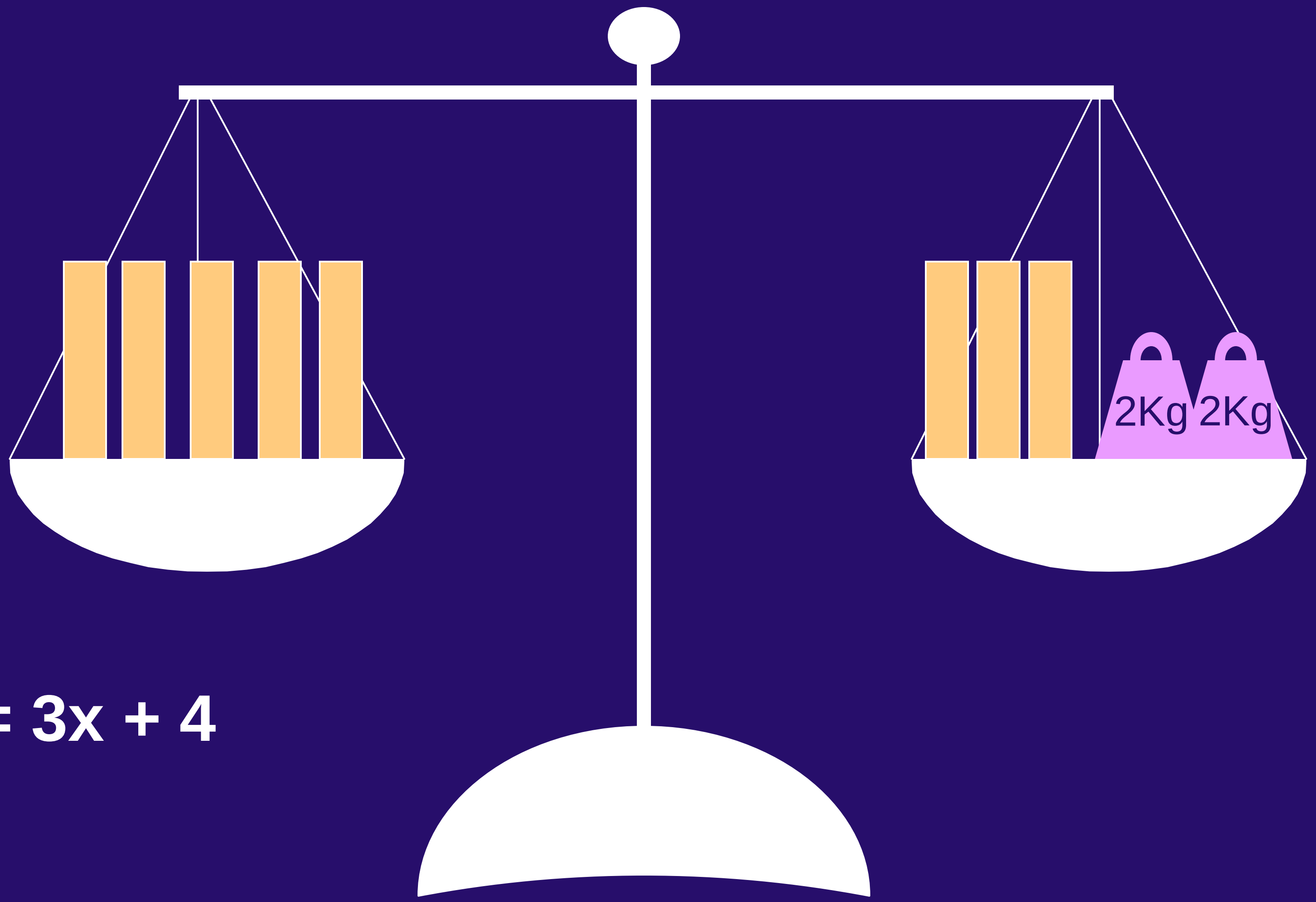




J. Porzio (2006)

Moving from Expressions to Equations

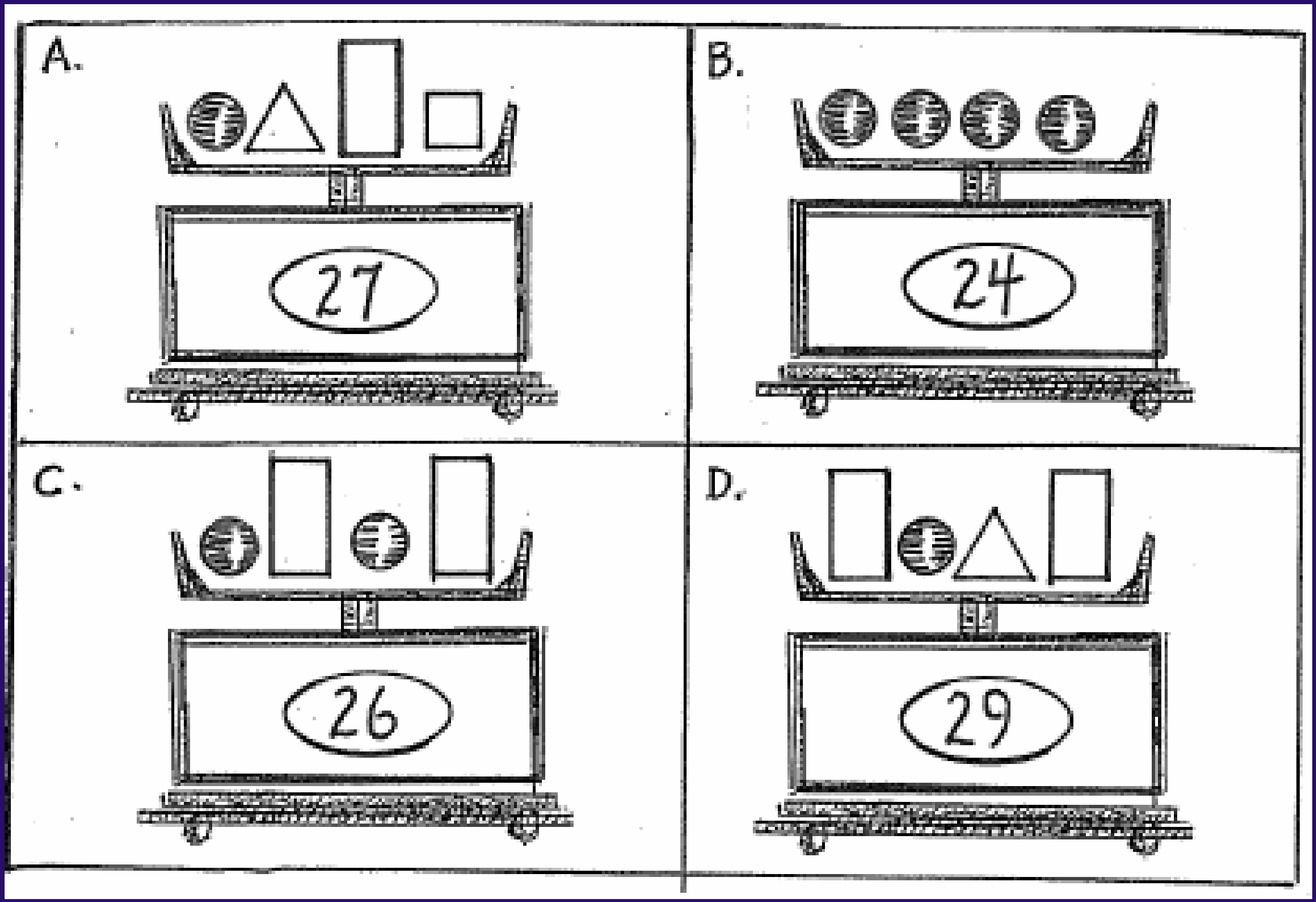
- Problem: The left pan of a set of scales contains 5 identical boxes of noodles, and the right pan contains 3 identical boxes and two 2-kg weights. The scales are balanced. How much does each box weigh?



$$5x = 3x + 4$$

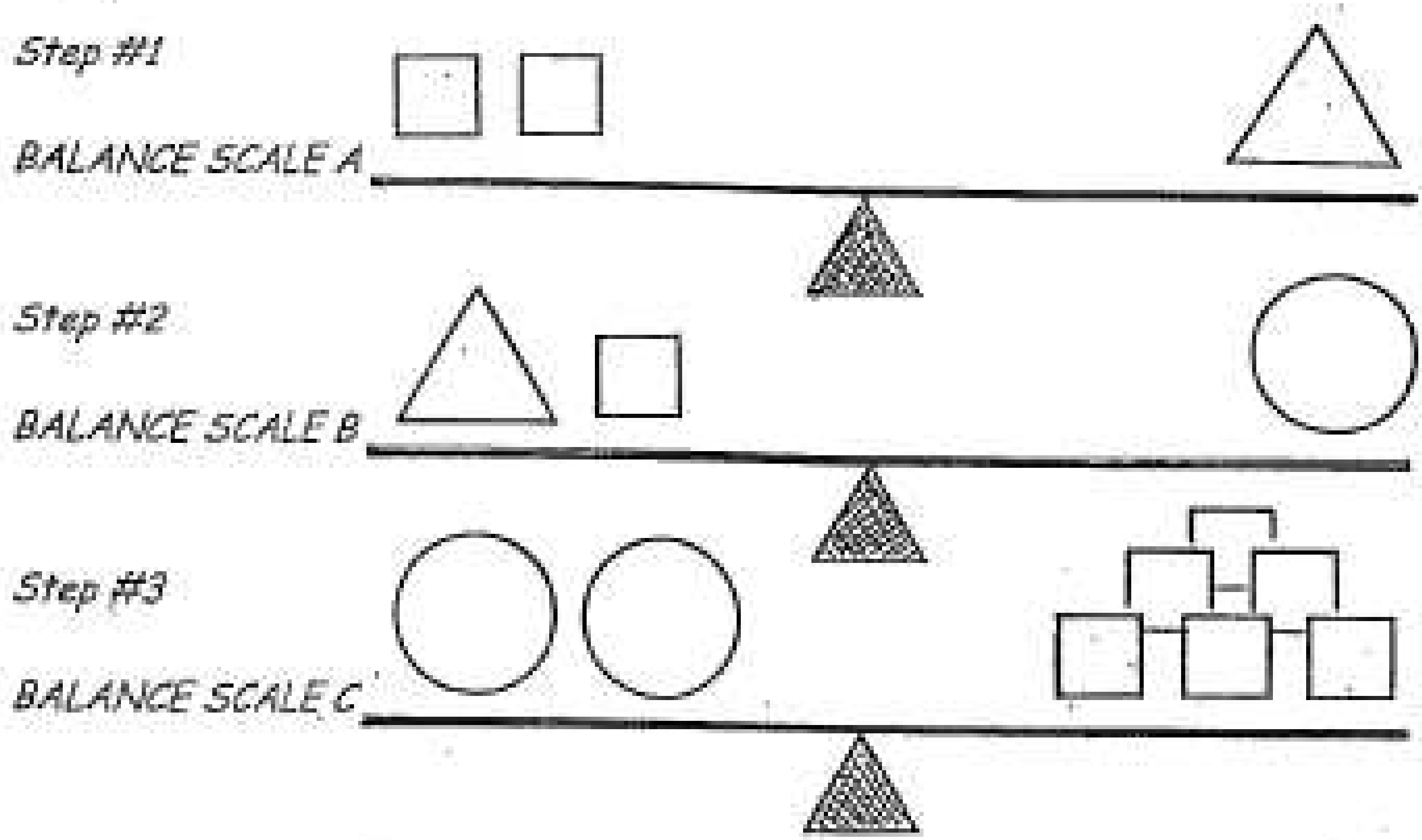
Equation: An equality with a variable.
Root: Each value of the variable that makes the statement a true equality.

The scale in each part shows the total weight of the shapes on that scale. The same shapes have the same weight in each of the pictures. Find the weight of each shape.
HINT: You may begin with any one of the four pictures that will help you start.



J. Porzio (2006)

Give your own values to the figures in A, B, and C. The <u>same</u> figures in A, B, and C will have the <u>same</u> values. <u>Different</u> figures will stand for <u>different</u> values. *Remember* to keep the scales in A, B, and C balanced!



J. Porzio (2006)

17. Expressions with a Variable

Problem. 5 truckloads of sand were delivered to upgrade a playlot and then another x truckloads were delivered. Each truck held 3 tons of sand. How many tons of sand were delivered to the playlot?

5 + *t* truckloads

Each truckload contained 3 tons of sand

The total amount of sand delivered is 3 X (5 + *t*)

3 (5 + *t*) is an ***expression with a variable***

If t is 2, than 3 X (5 + 2) = 21

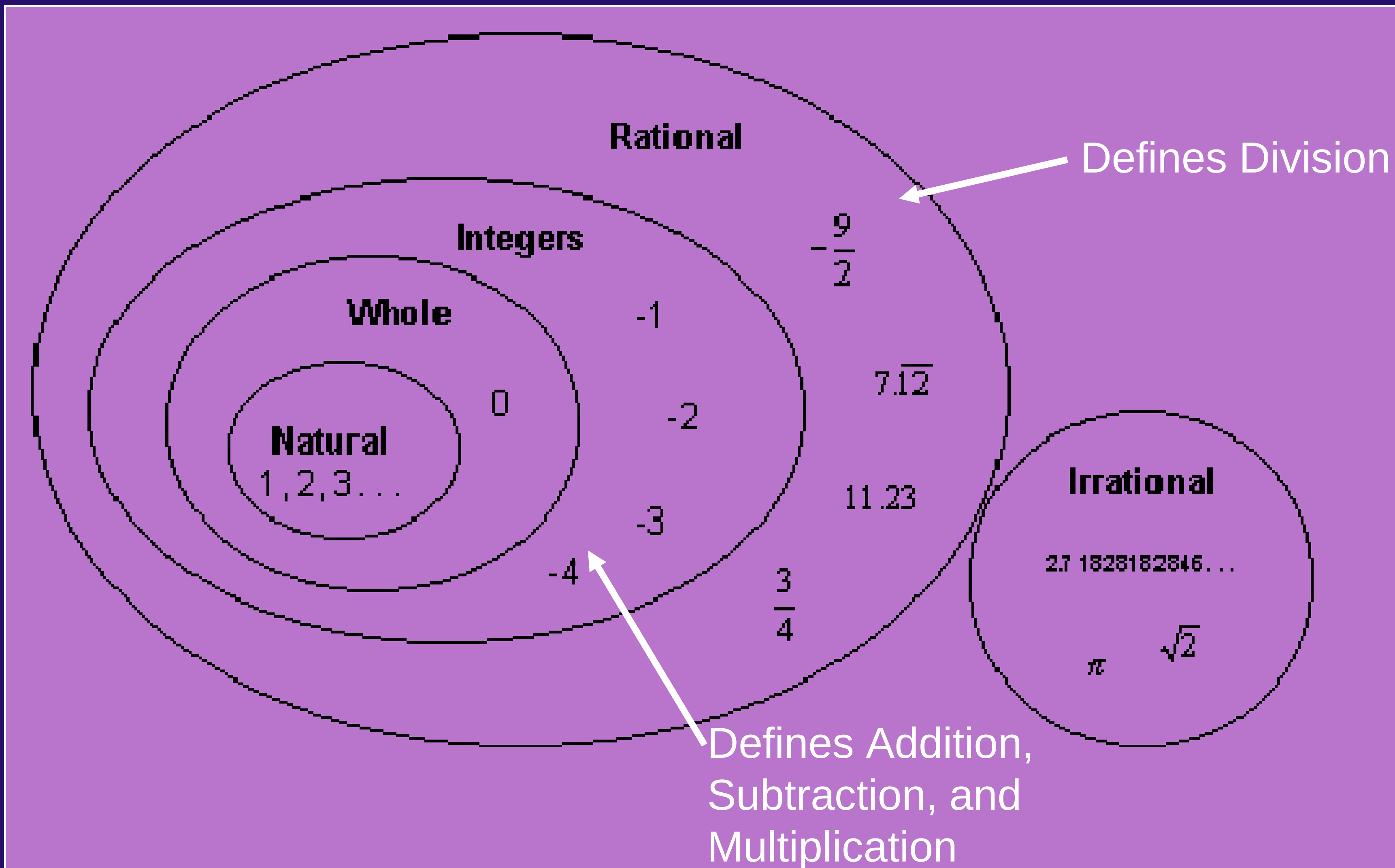
If t is 5, than 3 X (5 + 5) = 30

Milgram, 2005

Variables and Properties

- A variable represents a number–even though its value may not be given.
- Expressions with variables satisfy all the properties of the number system, such as the commutative, associative, and distributive properties.

The Real Numbers



$5 + 2 = 2 + 5$

Commutative Property of Addition

$5 + t = t + 5$

Commutative Property of Addition

$3 \times 7 = 7 \times 3$

Commutative Property of Multiplication

$3 \times t = t \times 3$

Commutative Property of Multiplication

$3 \times 7 \times 2 = 7 \times 2 \times 3$

Associative Property of Multiplication

$5 \times 3 \times t = t \times 3 \times 5$

Associative Property of Multiplication

$3 + 5 + 2 = 5 + 2 + 3$

Associative Property of Addition

$3 + 5 + t = 5 + t + 3$

Associative Property of Addition

$3 \text{ X } (5 + 2) = 3 \text{ X } 5 + 3 \text{ X } 2$

Distributive Property of Multiplication

$3 \text{ X } (5 + t) = 3 \text{ X } 5 + 3 \text{ X } t$

Distributive Property of Multiplication

t	0	1	7	9	20	95	900
$3 \text{ X } (5 + t)$	15	18	36	42	75	300	2715
$3 \text{ X } 5 + 3 \text{ X } t$	15	18	36	42	75	300	2715

Key Definitions: Expressions with More than One Variable

1. Solution of an equation: A value (or an ordered pair of values) that satisfies the equation
2. Equivalent equations: Equations that have the same solution set
3. Linear equation: An equation equivalent to one of the form $ax + by = c$ with $a^2 + b^2$ not equal to 0
4. Function: A rule connecting two sets that assigns to each element of one set (or input) one and only one element of the second set (or output)
5. Graph of an equation in two variables: Points in the plane whose coordinates satisfy the equation
6. Sequence: A function from the positive integers to the real numbers.

Expressions involving Two or More Variables…

- Adhere to the same commutative, associative, and distributive properties:

$$x(4x^3 + 2y^3) = x \cdot 4 \cdot x^3 + x \cdot 2 \cdot y^3$$

$$= 4 \cdot x \cdot x^3 + 2 \cdot x \cdot y^3$$

$$= 4x^4 + 2xy^3$$

A Historical Statement on Translating Word Problems

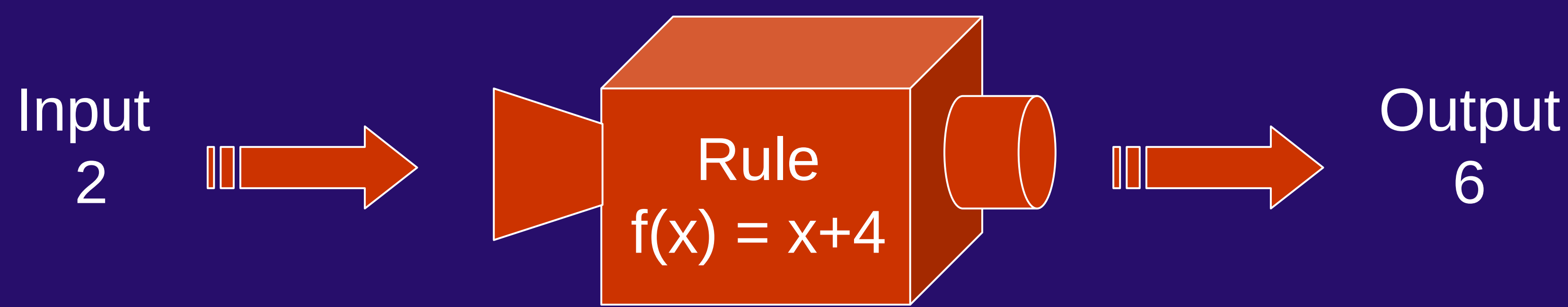
- In solving a word problem by setting up equations, the student translates a real situation into mathematical terms: he has an opportunity to experience that mathematical concepts may be related to realities, but such relations must be carefully worked out. Here is the first opportunity afforded by the curriculum for this basic experience.

G. Polya, *Mathematical Discovery*, Volume 1, P. 59

Process for Translating Word Problems to Algebra Problems

1. Students verbally read and explain what an expression/equation means,
2. Students formulate a verbal instruction as an algebraic expression,
3. Students translate components of word problems into mathematical expressions,
4. Students construct word problems associated with a particular algebraic expression
5. Students define variables, and
6. Students explicitly solve problems

Introduction to Functions



Functions Can Be Introduced Early without Formal Definitions

Definition

A rule between two sets such that there is a unique element in the second set assigned to each in the first set

Synonyms

rule of correspondence

function

$f(x) = x + 4$

$\{(1, 2), (3, 2), (5, 1)\}$

Examples

$3y + 5x$

$\{(1, 2), (1, 3), (4, 0)\}$

Counter Examples

Mathematics Progress Monitoring in Secondary Grades

Monitoring Student Progress

- The process of collecting and evaluating data to make decisions about the adequacy of student progress toward a goal and/or responding to instruction or interventions
- Evaluation of student rate of change (slope) as compared to the slope of anticipated progress
- Requires:
  - Technically sound measures
  - Multiple forms of the same measure
  - Assessment systems that are sensitive to student growth
  - Standardized administration procedures
  - Frequent measurement (occurs at least monthly)

Research Supports the Use of Progress Monitoring

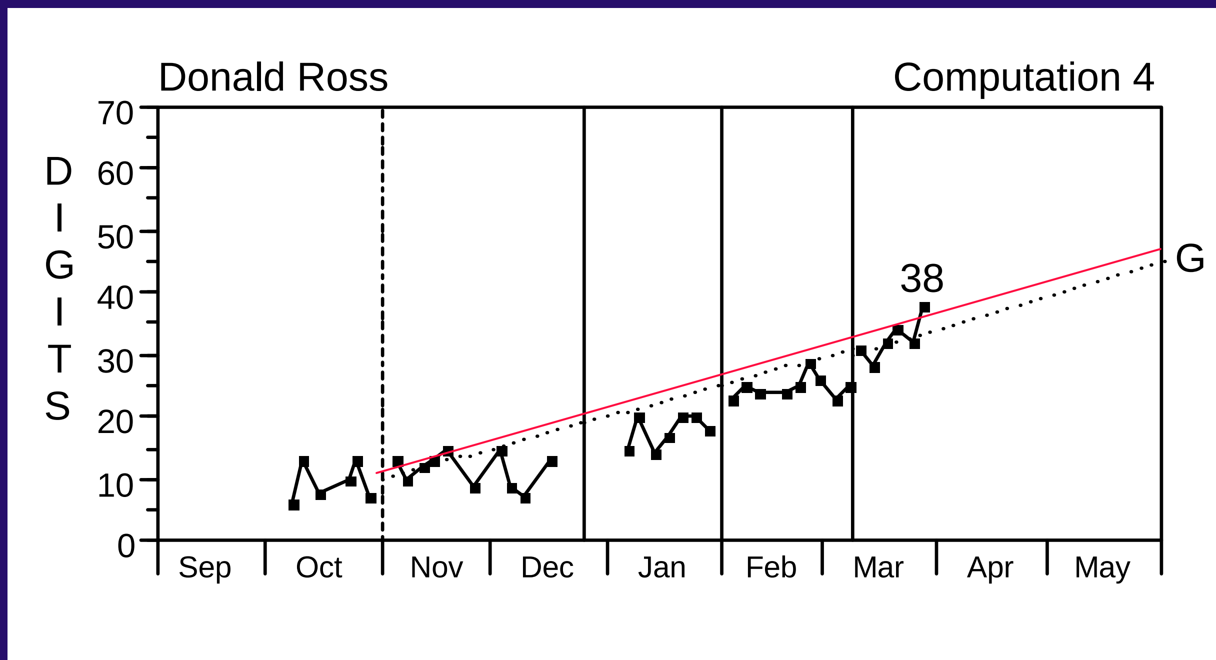
- Progress monitoring data produce accurate, meaningful information about students' academic levels and their rates of improvement
- Progress monitoring data are sensitive to student improvement
- Performance on progress monitoring measures corresponds well to performance on high-stakes tests
- When teachers use progress monitoring data to inform their instructional decisions, students make greater learning gains

Adapted from NCSPM

Process of Progress Monitoring

- Progress monitoring is a data-based instructional decision making tool
- Steps for using data:
  - Gather baseline performance data
  - Set instructional goals
  - Provide targeted instruction
  - Monitor progress toward goal
  - Adjust goal upward or modify instruction as needed

Example of Progress Monitoring Data



Adapted from NCSPM

Features of Progress Monitoring Systems

- Data are collected and evaluated frequently
  - Schedule is determined by goal and current level of student performance
  - Typically ranges from 2 times per week to monthly
- Teachers may choose to monitor progress of all students in class
- Typically, students at-risk of failure are assessed until they reach proficiency
- Data-based decision rules are applied to graphed data to determine when goals should be raised or instruction should be modified

Measuring Secondary Students' Progress in Mathematics

- Development of progress monitoring measures for secondary grades (especially beyond general math) is in its infancy
- Project AAIMS is developing and validating measures for pre-algebra and first year algebra
- Algebra measures have been created using both the robust-indicators and the curriculum-sampling methods

Algebra Progress Monitoring Research Results

- Project AAIMS evaluates technical adequacy and teachers’ use of algebra measures
- Reliability, criterion validity, and sensitivity to growth over time is documented for the Basic Skills, Algebra Foundations, and Content Analysis-Multiple Choice measures
- Less data on the Translations measure due to mismatch with existing curriculum materials in participating school districts
- Research is on-going to continue refinement of the measures

Examples of Algebra Progress Monitoring Measures

- Basic Skills
- Algebra Foundations
- Translations
- Content-Analysis-Multiple Choice

Basic Skills in Algebra

- Robust indicator of pre-algebra/algebra proficiency
- Problems include:
  - Solving basic “fact” equations
  - Applying the distributive property
  - Working with integers
  - Combining like terms
  - Simplifying expressions
  - Applying proportional reasoning
- Timed administration
- Constructed-response items
- Scored by counting number of problems correct

Basic Skills

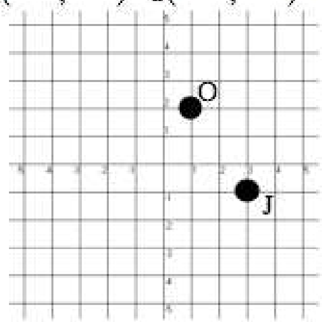
Solve: $9 + a = 15$ $a =$	Solve: $10 - 6 = g$ $g =$
Evaluate: $12 + (-8) + 3$	Simplify: $9 - 4d + 2 + 7d$
Simplify: $2x + 4 + 3x + 5$	Simplify: $5(b - 3) - b$
Solve: $12 - e = 4$ $e =$	Solve: $q \bullet 5 = 30$ $q =$
Simplify: $4(3 + s) - 7$	Evaluate: $8 - (-6) - 4$
Simplify: $b + b + 2b$	Simplify: $2 + w(w - 5)$
Solve: $\frac{b}{6} = \frac{12}{18}$ $b =$	Solve: 1 foot = 12 inches 5 feet = ____ inches
Simplify: $7 - 3(f - 2)$	Simplify: $4 - 7b + 5(b - 1)$
Evaluate: $-5 + (-4) - 1$	Simplify: $s + 2s - 4s$
Solve: $63 \div c = 9$ $c =$	Solve: $x + 4 = 7$ $x =$
Simplify: $2(s - 1) + 4 + 5s$	Simplify: $-5(q + 3) + 9$
Simplify: $8m - 9(m + 2)$	Evaluate: $9 + (-3) - 8$

Algebra Foundations

- Robust indicator of pre-algebra/algebra proficiency
- Problems include:
  - Writing and evaluating variables and expressions
  - Computing expression (integers, exponents, and order of operations)
  - Graphing expressions and linear equations
  - Solving 1-step equations and simplifying expressions
  - Identifying and extending patterns in data tables
- Timed administration
- Constructed-response items
- Scored by counting number of problems correct

Algebra Foundations

Find the ordered pair for each point:
J(,) O(,)



Fill in the empty box:

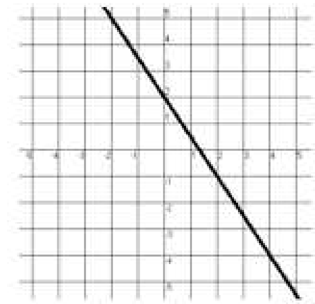
s	$3s$
6	18
7	21
8	
9	27

Fill in the empty box:

n	$4n + 7$
-1	3
-2	-1
-3	
-4	-9

Fill in the empty box:

b	
5	2
3	0
0	-3
-2	-5



What is the slope?

What is the y-intercept?

If $y > 9$, two possible values for y are _____ and _____

Evaluate:
$9 \bullet 4 - 6$

Simplify:
$7f + (2f + f)$

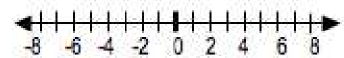
Solve:
$n + 3 = 8$
$n =$

Evaluate $4b + 2$ when
$b = 1$ ________ and when
$b = 3$ ________

Write the expression for this phrase:
6 less than a number

Evaluate:
$(-2) \bullet (-4)$

Graph the expression $m > 6$



Write a word phrase for this expression:
$n + 9$

Evaluate:
$4 + (9 + 3) - 2^2$

Evaluate:
$(-2)^3$

Write the expression for this phrase:
9 multiplied by a number

Evaluate $2x + y$ when
$x = 2$ and $y = 3$

Write a word phrase for this expression:
$10b - 7$

If $2a + 4 < 20$, two possible values for a are ________ and _______

Simplify:
$6 - 2(b - 4)$

Content Analysis

- Curriculum sampling approach to algebra proficiency
- Problems are sampled from core concepts in the first 2/3 of a traditional Algebra 1 textbook
- Multiple-choice items with partial credit
- Scored by counting number of points earned
  - Up to 3 points per problem awarded using a scoring rubric
  - -1 point penalty for circling an incorrect answer without showing any work (guessing)

Content Analysis - Multiple Choice

Evaluate $b^2 - a^2$ when $a = 4$ and $b = 5$

a) 21
b) 1
c) 11
d) 9

Rewrite this expression without parentheses:
(-5) (4 – y)

a) 9 - y
b) -20 + 5y
c) -1 – 5y
d) -20 – 5y

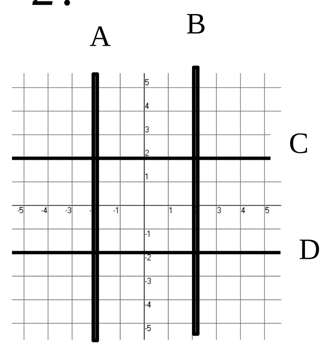
Solve:
$2t - 5 = 7$

a) $\frac{1}{2}$
b) 6
c) 1
d) 2

Solve:
$\frac{y}{3} = 4$

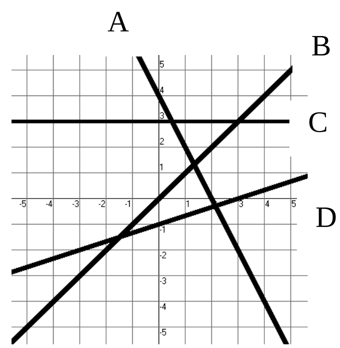
a) -10
b) 7
c) $\frac{4}{3}$
d) 12

Which line on the graph is
$y = 2$?



a) Line A
b) Line B
c) Line C
d) Line D

Which line on the graph is
$y + 2x = 4$?



a) Line A
b) Line B
c) Line C
d) Line D

Write the equation in slope-intercept form:
$m = (\frac{1}{2})$ $b = 3$

a) $y = 2x + 3$
b) $y = 3x + \frac{1}{2}$
c) $x = \frac{1}{2}y - 3$
d) $y = \frac{1}{2}x + 3$

Rewrite this equation in standard form using integer coefficients.
$-4y + \frac{1}{2}x = 2$

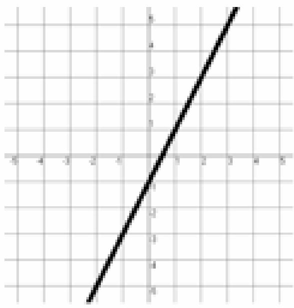
a) $-8y + 2x = 4$
b) $x - 8y = 4$
c) $y = 4x + 8$
d) $4y - 2x = 4$

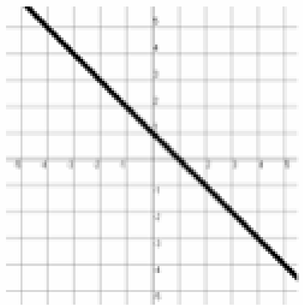
Translations

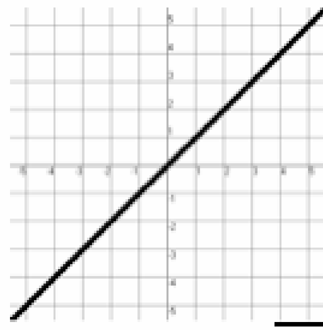
- Robust indicator of pre-algebra/algebra proficiency
- Problems include:
  - Tasks drawn from Connected Mathematics Project (CMP) Curriculum
  - Translate representations for relationships between two variables
    - Data tables
    - Graphs
    - Equations
    - Story scenarios
- Timed administration
- Scored by counting number of problems correct and subtracting number of problems incorrect

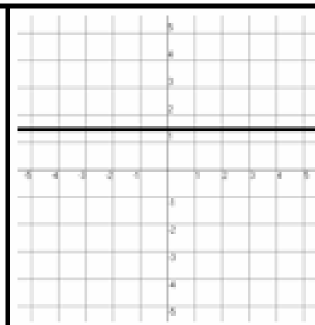
Translations

A	B	C	D
$y = x$	$y = 2x - 1$	$y = 1.5$	$y = -x + 1$









x	y
2	1.5
1	1.5
0	1.5
-1	1.5
-2	1.5

x	y
2	-1
1	0
0	1
-1	2
-2	3

x	y
2	3
1	1
0	-1
-1	-3
-2	-5

x	y
4	4
2	2
0	0
-2	-2
-4	-4

x	y
4	-3
2	-1
0	1
-2	3
-4	5

Mark needs to find half the width of pieces of pipe he is cutting to make a soccer goal. The width of the pipe is 3 inches. He wrote this equation to show the relationship between the length and the width of the pieces he will cut.	
Every day that Cindy waters the garden, she earns a dollar. She wrote this equation to show the relationship between the number of days she waters the garden and the number of dollars she will earn.	
Joe has one dollar in his wallet. He wrote this equation to show the relationship between the number of dollars he borrows from his friends for lunch and the total amount of money he has or owes.	
Mia earns $2 for each magazine subscription sold in the fund-raiser. A $1 fee per student is charged for a processing fee. Mia wrote this equation to show the relationship between the number of magazines sold and the profit.	
The flood waters are receding at a rate of 1 foot per day. The river is currently at 1 foot above flood stage. Tom wrote this equation to show the relationship between the number of days and the height of the river compared to flood stage.	

Summary: Uses of Progress Monitoring Data

- Estimate rates of student improvement
- Describe student response to instructional program
- Inform teachers’ instructional decision making
- Aid teachers in targeting areas/skills that need remediation
- Help teachers build potentially more effective programs for particular students

References

Text Material:

- Milgram, J. (2005). XXXX
- Porzio, J. (2006). XXXX

Websites:

- Project AAIMS Web site
  www.ci.hs.iastate.edu/aaims

Additional Resources

National Centers

- National Center on Student Progress Monitoring (NCSPM): http://www.studentprogress.org
- Research Institute on Progress Monitoring (RIPM): http://www.progressmonitoring.org

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